The Vacuum Fluctuation Contribution of an Ultra Light Scalar Field in the Presence of Extra Dimensions

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Experimental data obtained from CMB measurements and SNIa supernova observations indicate that the effective energy density of the universe is of order $10^{-48}(GeV)^4$. Theoretical evaluations of the vacuum energy with a Planck scale cutoff predicts a far higher energy density of order $10^{71}(GeV)^4$. This catastrophic discrepancy between observation and theory has been called the 'worst prediction in the history of physics'. In this paper we present the cosmological problem and demonstrate how a 'toy' universe filled with an ultra light scalar field in the presence of a compact extra dimension can account for the observed cosmological constant using a ζ function regularization scheme.

I. INTRODUCTION

The search for the nature of the dark energy component of our universe has ignited a fury of speculation. It can be described as a smooth component with wavelength of the order of the diameter of the universe, which contributes at least 70% of the energy density in the universe [1].

In this letter we explore how an ultra light scalar field in the presence of an extra dimension can be used to recover the observed value of the cosmological constant.

II. THE COSMOLOGICAL CONSTANT AND VACUUM ENERGY

Since the universe looks spatially homogeneous at large distance scales, it is appropriately described by the Robertson-Walker metric

$$ds^2 = -dt^2 + a(t)^2 R_0^2 \left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right] \tag{1}$$

where $d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$ is the metric on a two sphere, $a(t) = \frac{R(t)}{R_0}$ is a scale factor where R_0 is the current scale length of the universe and k is the curvature parameter that takes values of +1 for positive curvature, 0 for flat and -1 for negatively curved space.

One of the fundamental premises of building cosmological models is considering the space to be filled with an isotropic perfect fluid of density ρ and pressure p. The energy momentum tensor of such a fluid is

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} \tag{2}$$

where u_μ is the velocity measured in a frame comoving with the fluid. Having provided a description of the matter content and geometry of the spacetime considered, the Einstein field equations can now be employed

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{3}$$

For the simplest case of a static universe ($\dot{a} = 0$) with $k = 1$, the Einstein equations reduce to the Friedmann equations

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2 R_0^2} \tag{4}$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{5}$$

For normal matter with $p > 0$ these equations imply that the universe will accelerate forever, having a positive $\ddot{a}$. When Einstein arrived at this solution he was perturbed, believing that the dynamics of the universe as predicted by General Relativity, should be static in accordance to Mach's principle [2]. He thus suggested a modification to his field equations of the form

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{6}$$

where Λ, the cosmological constant is still a free parameter at this point. This modification changes the Friedmann solutions to

$$H^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2 R_0^2} \tag{7}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \tag{8}$$

This solution is called the "Einstein static universe", that permits a static solution with positive ρ, p and Λ.

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Hubble's discovery of an expanding universe [3] negated the need for this ad-hoc term, but the spirit of this modification as a free-parameter never faded. Particle physicists have taken advantage of the fact that the matter density in the universe can still not be measured to the precision where the $\frac{\Lambda}{3}$ term becomes negligible and have considered it to be the vacuum energy.

If we define a scalar field with potential energy $V(\phi)$ we can write its energy momentum tensor as

$$T_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}\left(g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi\right)g_{\mu\nu} - V(\phi)g_{\mu\nu} \tag{9}$$

If we were to assume that it permeates our universe, then the lowest energy state for this field would correspond to the energy of the vacuum. Thus the minimized state results from vanishing kinetic and gradient terms ($\partial_\mu = 0$) and results in

$$T_{\mu\nu} = -V(\phi_0)g_{\mu\nu} \tag{10}$$

where ϕ_0 is the value that minimizes the potential to the ground state of the vacuum, that we need not consider to be zero. Thus

$$V(\phi_0) = \rho_{vac} \tag{11}$$

and

$$T^{vac}_{\mu\nu} = -\rho_{vac}g_{\mu\nu} \tag{12}$$

We are at liberty to consider the vacuum to be a perfect fluid with equation of state

$$p_{vac} = -\rho_{vac} \tag{13}$$

If we move the cosmological constant term in (6) to the right hand side of the Einstein equation, we can now account for the combined contributions of the free space and vacuum energy momentum tensors

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G\sum_i T^i_{\mu\nu} \tag{14}$$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{\Lambda}{8\pi G}g_{\mu\nu}\right) \tag{15}$$

Comparing with (15) to (12) we can see that the cosmological constant can be thought of as the vacuum energy:

$$\rho_{vac} = \rho_\Lambda = \frac{\Lambda}{8\pi G} \tag{16}$$

Efforts have been made to use quantum field theory to explain, on a more fundamental level, both the origin and the value of the vacuum energy.

Discarding high momentum modes beyond the confidence level of our theory we can get a prediction of the resulting energy density indicated from this ultraviolet cutoff, from the equation

$$\rho_\Lambda \sim k^4_{max} \tag{17}$$

since $E \sim k$ and $L^3 \sim E^{-3}$ by dimensional analysis, and $\hbar = 1$ in natural units.

It is reasonable to use the results from the successful Lamb Shift and Casimir Effect theories from quantum electrodynamics to explore the value of vacuum fluctuations [4]. Using the Plank mass ($\sim 10^{19}GeV$) as the natural UV cutoff in General Relativity, we find the quadratically divergent energy density to be

$$\rho_{Plank} = (10^{19}GeV)^4 = 10^{112}(eV)^4 \tag{18}$$

If we use the electroweak symmetry breaking scale ($\sim$ 100 GeV) as our cut-off, we get

$$\rho_{EW} = (10^2GeV)^4 = 10^{44}(eV)^4 \tag{19}$$

These seemingly appropriate energy cut-off scales give predictions for the vacuum energy density that are far in excess of the measured value.

$$\rho_\Lambda = (10^{-12}GeV)^4 = 10^{-48}(GeV)^4 \tag{20}$$

Predicting an accurate value for the vacuum energy density has thus provided physicists with a fascinating and tantalizing challenge.

III. VACUUM ENERGY CALCULATIONS

So far this paper has been concerned with classical theories regarding the cosmological constant. In this section we shift into the language of quantum field theory and demonstrate how the ground state vacuum fluctuations of a scalar field in the presence of a compact extra dimension, that can be made to account for the overall energy density of the universe.

As the most general case we will assume the universe is filled with a hypothetical light scalar field with mass m_ϕ. The total energy of the ground state fluctuations of the field in the presence of an extra dimension is

$$E_\phi = -\int \frac{d^3k}{(2\pi)^3}\sqrt{k^2 + \frac{n^2}{R^2} + m_\phi^2} \tag{21}$$

where R is the radius of the extra dimension and n are the Kaluza Klein excitation modes. Note that since we seek the ground state of the vacuum, we do not have divergent terms that require some UV cuttoff.

Following the techniques of [5] we perform a ζ-function regularization and introduce an additional mass parameter μ. The the purpose of the mass parameter μ is to restore the correct dimension for the regularized terms.

$$E_\phi(\epsilon) = -\mu^{2\epsilon} \int \frac{d^3k}{(2\pi)^3} \frac{1}{(k^2 + \frac{n^2}{R^2} + m_\phi^2)^{\epsilon - \frac{1}{2}}} \tag{22}$$

Where for $\epsilon \to 0$ we recover equation (20). Performing the integral we obtain

$$E_\phi(\epsilon) = -\frac{\left(m_\phi^2 + \frac{n^2}{R^2}\right)^2}{8\pi^{3/2}} \left(\frac{\mu^2}{m_\phi^2}\right)^\epsilon \frac{\Gamma(\epsilon - 2)}{\Gamma(\epsilon - \frac{1}{2})} \tag{23}$$

Using well understood Γ-function relations and performing the appropriate expansions for small ϵ the energy becomes

$$E_\phi(\epsilon) = -\frac{\left(m_\phi^2 + \frac{n^2}{R^2}\right)^2}{32\pi^2} \left(\frac{1}{\epsilon} + 2ln2 - \frac{1}{2}\right) ln\frac{\mu^2}{m_\phi^2 + \frac{n^2}{R^2}} + O(\epsilon) \tag{24}$$

As $\epsilon \to \infty$ the divergent quantity can be extracted allowing for the familiar constant Λ to be redefined as

$$\Lambda \to \Lambda_0 - \frac{1}{\epsilon}\frac{m^2 + \frac{n^2}{R^2}}{32\pi^2} \tag{25}$$

We are thus left with an expression for the effective energy density

$$\rho_{eff} = \frac{\Lambda_0}{8\pi G} + \frac{\left(m_\phi^2 + \frac{n^2}{R^2}\right)^2}{32\pi^2} \left(ln\frac{\mu^2}{m_\phi^2} + 2ln2 - \frac{1}{2}\right) \tag{26}$$

ρ_{eff} is clearly a function of the mass scale μ and must remain unchanged under variations of the parameter. Following the work of [6] and [7] we arrive at the renormalization group equation

$$\mu\frac{d\rho_{eff}}{d\mu} = \frac{\mu}{8\pi G}\frac{d\Lambda_0(\mu)}{d\mu} + \frac{\left(m_\phi^2 + \frac{n^2}{R^2}\right)^2}{16\pi^2} = 0 \tag{27}$$

Solving this equation illustrates that Λ_0 varies with μ as

$$\Lambda_0(\mu_0) = -\frac{G\left(m_\phi^2 + \frac{n^2}{R^2}\right)^2}{2\pi} \; ln\frac{\mu}{\mu_0} \tag{28}$$

where

$$\Lambda_0(\mu_0) = 0 \tag{29}$$

Substituting (28) into (26) we arrive at the equation

$$\rho_{eff} = \frac{\left(m_\phi^2 + \frac{n^2}{R^2}\right)^2}{16\pi^2} \left(ln\frac{\mu_0}{m_\phi} + 2ln2 - \frac{1}{2}\right) \tag{30}$$

We can approach the solution to this equation in two ways. For simplicity lets first assume that the extra dimension is in its lowest KK excitation state, $n = 0$, μ_0 is on the order of the Planck scale, $10^{19} GeV$ and that the scalar field is ultra-light with mass m_ϕ of order $10^{-12} GeV$.

With the assigned value of the scalar field we can recover the effective renormalized cosmological constant which is in agreement with current observations.

$$\rho_{eff} \sim 10^{-48}\ GeV^4 \tag{31}$$

If we now solve equation (30) assuming that the contributions to ρ_{eff} come entirely from the $n = 1$ Kaluza-Klein excitation, and we use the observed value for ρ_{eff}, we can estimate the radius of the extra dimension

$$R \sim 10^{12}\ GeV^{-1} \tag{32}$$

To recover the length scale of the extra dimension we use the conversion 1 m = $5.07 \times 10^{15}\ GeV^{-1}$, which results in

$$R \sim 10^{-4}\ m \tag{33}$$

We find the size of the extra dimension to be of the order required so that it remain hidden by particle accelerator experiments and observed astrophysical processes.

IV. CONCLUSION

We have shown how a 'toy' universe with a single compact extra dimension, can recover the observed value of the cosmological constant. The method of ζ-function regularization shown here for a very simple case demonstrates the capability to recover realistic results. For more sophisticated approaches and details on this method please refer to [8].

In summary, by filling the universe with an ultra light scalar field, that we identify would have to have mass m_ϕ of order 10^{-12} GeV, we can recover the currently observed value of the cosmological constant. No such scalar field is known to exist, however, the closest physical candidate being a free neutrino field with mass m_ν of order 10^{-9} GeV. Using the neutrino field a value of $\rho_{eff} = 10^{-36}\ GeV^4$ is obtained [5]. Though still many orders of magnitude above the observed value, it is much closer than the catastrophic predictions mentioned earlier.

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